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# Positive solutions for a Floquet functional boundary value problem

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**Abstract**

In this paper we prove the existence of multiple positive solutions for a boundary value problem concerning a first order functional differential equation. The results are obtained by using two fixed point theorems on appropriate cones in Banach spaces. These theorems are based on the fixed point index theory.



*Keywords:* First order; Multiple positive solutions; Floquet boundary value problems; Functional differential equations

## 1. Introduction

Let $\mathbb{R}$ be the set of real numbers. If $B$ is a compact subset of $\mathbb{R}$, we denote by $C(B)$ the Banach space of all continuous real functions $\psi : B \to \mathbb{R}$ endowed with the usual sup-norm

$$\|\psi\|_B := \sup\{|\psi(s)|: s \in B\}.$$

In the following we consider the intervals $I := [0, T]$ and $J := [-r, 0]$, where $T, r$ are real numbers with $0 \leqslant r < T$. For any continuous function $x \in C(J \cup I)$ and any $t \in I$, we denote by $x_t$ the element of $C(J)$ defined by

$$x_t(s) = x(t + s), \quad s \in J.$$

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Now, consider the equation

$$x'(t) = f(t, x_t), \quad t \in I, \tag{1.1}$$

with the boundary condition

$$Ax_0 - x_T = \phi, \tag{1.2}$$

where $f : I \times C(J) \to \mathbb{R}$, $\phi : J \to \mathbb{R}$ are continuous functions and it holds that

($H_1$) $A > 1$, $\phi(0) \geqslant 0$ and $\phi(t) \geqslant -\phi(0)/(A-1)$, $t \in J$.

For a detailed review on the class of functional differential equations of the type (1.1), we refer to the books by Hale and Lunel [17], Azbelev et al. [8] and Driver [12].

The boundary value problem (1.1)–(1.2) belongs to a class of problems known as "Floquet problems" (see [23,26]). These problems arise from physics. A linear version of this problem was early studied by Cooke in [10]. Also, Conti in [9] discusses, among other boundary value problems, an ordinary analogue of (1.1)–(1.2). This is also the case in [23], where the existence of solutions is based on $f$ being sublinear. However functional boundary value problems similar to (1.1)–(1.2) are the ones dealt with in [13] and [31]. Here the so called "shooting method" is used to determine the existence of at least one solution. Similar boundary value problems, but dealt with various other methods can be found in [15,16,25,27–30]. Additionally, in [6], a less similar problem for first order implicit ordinary differential equations is treated using the measure of noncompactness.

In the recent years, an increasing interest has been observed in studying the existence of positive solutions for boundary value problems. The book by Agarwal et al. [4] gives a good overview on this issue. This is widely done for second order boundary value problems, usually using the well known Krasnoselskii's fixed point theorem. That is due to the fact that in the case of second order problems, the solutions are usually convex, so this theorem can be easily applied. The reader could refer among others to [2,7,18,20,21,24] and references therein for some recent papers providing relative results.

However, the interest for investigating analogous problems involving functional boundary value problems, arose soon. Some papers dealing with such problems are [19,22,33].

In this paper we will use two fixed point theorems, that can be found in [5] and [11], to show that the boundary value problem (1.1)–(1.2) has (at least) one positive solution, which is upper and lower bounded by specific real numbers. These theorems are based on the theory of fixed point index and for an extended overview, the reader should refer to [5,11,14,32]. We will also show how we can obtain multiple positive solutions by repeatedly applying the aforementioned theorems.

The paper is organized as follows. Section 2 contains the basic preliminaries, the fixed point theorems that we are going to use and the reduction of the problem (1.1)–(1.2) to an abstract operator equation. The main results are given in Sections 3 and 4. The results in Section 3 concern the functional boundary value problem (1.1)–(1.2) and the results in Section 4 concern the corresponding ordinary boundary value problem, which is derived from (1.1)–(1.2), when $r = 0$. This is done because the results are new in the ordinary case too. In Section 5 some applications are presented.

## 2. Preliminaries and some basic lemmas

**Definition 2.1.** Let $\mathbb{E}$ be a real Banach space. A cone in $\mathbb{E}$ is a nonempty, closed set $\mathbb{P} \subset \mathbb{E}$ such that

(i) $\kappa u + \lambda v \in \mathbb{P}$ for all $u, v \in \mathbb{P}$ and all $\kappa, \lambda \geqslant 0$,
(ii) $u, -u \in \mathbb{P}$ implies $u = 0$.

Let $\mathbb{P}$ be a cone in a Banach space $\mathbb{E}$. Then, for any $b > 0$, we denote by $\mathbb{P}_b$ the set

$$\mathbb{P}_b = \{x \in \mathbb{P}\colon \|x\| < b\}$$

and by $\partial\mathbb{P}_b$ the boundary of $\mathbb{P}_b$ in $\mathbb{P}$, i.e., the set

$$\partial\mathbb{P}_b := \{x \in \mathbb{P}\colon \|x\| = b\}.$$

In order to prove our results, and since we are looking for positive solutions, we will use the following two theorems, which are applications of the fixed point theory in a cone. Their proofs can be found in [5,11].

**Theorem 2.2.** *Let $g : \bar{\mathbb{P}}_b \to \mathbb{P}$ be a completely continuous map such that $g(x) \neq \lambda x$ for all $x \in \partial\mathbb{P}_b$ and $\lambda \geqslant 1$. Then $g$ has a fixed point in $\mathbb{P}_b$.*

**Theorem 2.3.** *Let $g : \bar{\mathbb{P}}_b \to \mathbb{P}$ be a completely continuous map and $\sigma, \tau \in (0, b]$ with $\sigma \neq \tau$. Suppose that*

(i) *$g(x) \neq \lambda x$ for every $x \in \partial\mathbb{P}_\sigma$ and $\lambda \geqslant 1$,*
(ii) *there exists an element $p > 0$ such that $x - g(x) \neq \lambda p$ for every $x \in \partial\mathbb{P}_\tau$ and $\lambda \geqslant 0$.*

*Then $g$ has at least one fixed point $x$ with*

$$\min\{\sigma, \tau\} < \|x\| < \max\{\sigma, \tau\}.$$

The following Theorem 2.4 is an, easier to use, corollary of Theorem 2.3 and its proof can be found in [1,3].

**Theorem 2.4.** *Let $E = (E, \|\cdot\|)$ be a Banach space, $\mathbb{P} \subset E$ be a cone, and $\|\cdot\|$ be increasing (strictly) with respect to $\mathbb{P}$. Also, $\sigma, \tau$ are positive constants with $\sigma \neq \tau$. Suppose $g : \bar{\mathbb{P}}_{\max\{\sigma,\tau\}} \to \mathbb{P}$ is a completely continuous map and assume the conditions*

(i) *$g(x) \neq \lambda x$ for every $x \in \partial\mathbb{P}_\sigma$ and $\lambda \geqslant 1$,*
(ii) *$\|g(x)\| \geqslant \|x\|$ for $x \in \partial\mathbb{P}_\tau$*

*hold. Then $g$ has at least a fixed point $x$ with*

$$\min\{\sigma, \tau\} < \|x\| < \max\{\sigma, \tau\}.$$

**Definition 2.5.** A solution of the boundary value problem (1.1)–(1.2) is a function $x \in C(J \cup I)$ continuously differentiable on $I$, which satisfies Eqs. (1.1) and (1.2). Additionally, $x$ is called positive solution if $x(t) \geqslant 0$, $t \in J \cup I$.

In order to apply the above theorems we must reformulate our boundary value problem (1.1)–(1.2) into an abstract operator equation. This is done in the following lemma.

**Lemma 2.6.** *A function $x \in C(J \cup I)$ is a solution of the boundary value problem* (1.1)–(1.2) *if and only if $x(t) = Lx(t)$, $t \in J \cup I$, where $L : C(J \cup I) \to C(J \cup I)$ is given by the formula*

$$Lx(t) = \begin{cases} \frac{\phi(0)}{A-1} + \frac{1}{A-1}\int_0^T f(\theta, x_\theta)\,d\theta + \int_0^t f(\theta, x_\theta)\,d\theta, & t \in I, \\ \frac{\phi(0)}{A(A-1)} + \frac{1}{A(A-1)}\int_0^T f(\theta, x_\theta)\,d\theta + \frac{1}{A}\int_0^{T+t} f(\theta, x_\theta)\,d\theta + \frac{\phi(t)}{A}, & t \in J. \end{cases}$$

**Proof.** First observe that for every $x \in C(J \cup I)$ we have

$$\begin{aligned} Lx(0^-) &= \frac{\phi(0)}{A(A-1)} + \frac{1}{A(A-1)}\int_0^T f(\theta, x_\theta)\,d\theta + \frac{1}{A}\int_0^T f(\theta, x_\theta)\,d\theta + \frac{\phi(0)}{A} \\ &= \frac{\phi(0)}{A-1} + \frac{1}{A-1}\int_0^T f(\theta, x_\theta)\,d\theta = Lx(0^+). \end{aligned}$$

So, $Lx$ is a continuous function for every $x \in C(J \cup I)$. Moreover, from (1.1) we have

$$x(t) = x(0) + \int_0^t f(\theta, x_\theta)\,d\theta, \quad t \in I. \tag{2.1}$$

Since $r < T$, we observe that, if $s \in J$, then $T + s \in I$. Thus from (1.2) and (2.1) we get

$$Ax(s) - \left(x(0) + \int_0^{T+s} f(\theta, x_\theta)\,d\theta\right) = \phi(s), \quad s \in J. \tag{2.2}$$

Therefore, for $s = 0$, from Eq. (2.2) we have

$$Ax(0) - x(0) - \int_0^T f(\theta, x_\theta)\,d\theta = \phi(0)$$

or

$$x(0) = \frac{\phi(0)}{A-1} + \frac{1}{A-1}\int_0^T f(\theta, x_\theta)\,d\theta. \tag{2.3}$$

Using (2.1) and (2.3) we have

$$x(t) = \frac{\phi(0)}{A-1} + \frac{1}{A-1}\int_0^T f(\theta, x_\theta)\,d\theta + \int_0^t f(\theta, x_\theta)\,d\theta, \quad t \in I.$$

Also, combining (2.2) and (2.3) for $s \in J$ we get

$$x(s) = \frac{1}{A}x(0) + \frac{1}{A}\int_0^{T+s} f(\theta, x_\theta)\,d\theta + \frac{\phi(s)}{A}$$
$$= \frac{\phi(0)}{A(A-1)} + \frac{1}{A(A-1)}\int_0^{T} f(\theta, x_\theta)\,d\theta + \frac{1}{A}\int_0^{T+s} f(\theta, x_\theta)\,d\theta + \frac{\phi(s)}{A}.$$

So,

$$x(t) = Lx(t), \quad t \in [-r, T].$$

On the other hand, if $x \in C(J \cup I)$ is such that $x(t) = Lx(t)$, $t \in J \cup I$, then, it is clear that, for every $t \in I$ we have

$$x'(t) = \bigl(Lx(t)\bigr)' = f(t, x_t).$$

Also for any $s \in J$ we have

$$\begin{aligned} Ax_0(s) - x_T(s) &= Ax(s) - x(T+s) \\ &= \frac{\phi(0)}{A-1} + \frac{1}{A-1}\int_0^{T} f(\theta, x_\theta)\,d\theta + \int_0^{T+s} f(\theta, x_\theta)\,d\theta \\ &\quad + \phi(s) - \frac{\phi(0)}{A-1} - \frac{1}{A-1}\int_0^{T} f(\theta, x_\theta)\,d\theta - \int_0^{T+s} f(\theta, x_\theta)\,d\theta \\ &= \phi(s). \end{aligned}$$

The proof is complete. □

## 3. Positive solutions for the functional boundary value problem (1.1)–(1.2)

We set

$$C^+(J) := \bigl\{x \in C(J):\ x(t) \geqslant 0,\ t \in J\bigr\}.$$

The following assumptions are adopted throughout this section:

($H_2$) Assume that $f(I \times C^+(J)) \subset [0, +\infty)$ and for every $t \in I$ the function $f(t, \cdot)$: $C^+(J) \to [0, +\infty)$ maps bounded subsets of $C^+(J)$ into bounded subsets of $[0, \infty)$.

It is clear that under assumption ($H_2$), for every $s \in I$, $y \in C^+(J)$ and $m > 0$, $\sup_{\|y\|_J \in [0,m]} f(s, y)$ exists in $\mathbb{R}$. Then we set

$$F(s, m) := \sup_{\|y\|_J \in [0,m]} f(s, y).$$

Also for any $m > 0$ we set

$$\Theta(m) := \max_{t \in I} \int_0^t F(s, m)\, ds = \int_0^T F(s, m)\, ds$$

and

$$Q(m) := \max\left\{ \frac{\phi(0) + A\Theta(m)}{A - 1}, \frac{\phi(0) + A\Theta(m)}{A(A - 1)} + \frac{\|\phi\|_J}{A} \right\}.$$

($H_3$) Assume that there exists $\rho > 0$ such that $Q(\rho) < \rho$.

Additionally, we set

$$\Phi := \frac{\phi(0)}{A(A - 1)}$$

and we are ready to prove our first result.

**Theorem 3.1.** *Suppose that conditions* ($H_1$)–($H_3$) *hold. Also suppose that if* $\phi = 0$, *there exists* $t_0 \in I$ *such that* $f(t_0, 0) \neq 0$. *Then the boundary value problem* (1.1)–(1.2) *has at least one positive nonzero solution* $x$, *such that*

$$\Phi \leqslant \|x\|_{J \cup I} < \rho.$$

*More precisely we have*

$$x(t) \geqslant \Phi, \quad t \in J \cup I.$$

**Proof.** Let us set

$$\mathbb{P} = \big\{ x \in C(J \cup I) \colon x(t) \geqslant 0 \big\}$$

and observe that $\mathbb{P}$ is a cone in $C(J \cup I)$. It is also clear that for every $x \in \mathbb{P}$ and $t \in I$ we have $x_t \in C^+(J)$. Then by ($H_2$) we have $f(t, x_t) \geqslant 0$ and, taking into account ($H_1$), we easily obtain $Lx(t) \geqslant 0$ (the formula of $L$ is given in Lemma 2.6). This means that $L(\mathbb{P}) \subset \mathbb{P}$. So, since we are looking for a positive solution of the boundary value problem (1.1)–(1.2), it is enough to find a fixed point of the operator $L : \mathbb{P} \to \mathbb{P}$.

Let $x \in \bar{\mathbb{P}}_\rho$, where $\rho$ is the constant introduced by ($H_3$). Then, it is obvious that $L(\bar{\mathbb{P}}_\rho) \subset \mathbb{P}$. Also from ($H_2$) it follows that $L$ maps bounded subsets of $\bar{\mathbb{P}}_\rho$ into bounded subsets of $\mathbb{P}$, so $L : \bar{\mathbb{P}}_\rho \to \mathbb{P}$ is a completely continuous operator.

Furthermore, we will show that $\lambda x \neq Lx$ for every $\lambda \geqslant 1$ and $x \in \partial\mathbb{P}_\rho$. So let $x \in \partial\mathbb{P}_\rho$ and $\lambda \geqslant 1$ such that $\lambda x = Lx$. Then for every $t \in I$ we have

$$\begin{aligned}
|x(t)| \leqslant \lambda |x(t)| &\leqslant \frac{\phi(0)}{A - 1} + \frac{1}{A - 1} \int_0^T F(\theta, \rho)\, d\theta + \int_0^t F(\theta, \rho)\, d\theta \\
&\leqslant \frac{\phi(0)}{A - 1} + \frac{1}{A - 1}\Theta(\rho) + \Theta(\rho) = \frac{\phi(0) + A\Theta(\rho)}{A - 1}.
\end{aligned}$$

Also, for every $t \in J$ we have

$$\begin{aligned}
|x(t)| &\leqslant \lambda |x(t)| \\
&\leqslant \frac{\phi(0)}{A(A-1)} + \frac{1}{A(A-1)} \int_0^T F(\theta, \rho)\, d\theta + \frac{1}{A} \int_0^{T+t} F(\theta, \rho)\, d\theta + \frac{\phi(t)}{A} \\
&\leqslant \frac{\phi(0)}{A(A-1)} + \frac{1}{A(A-1)} \Theta(\rho) + \frac{1}{A} \Theta(\rho) + \frac{\|\phi\|_J}{A} \\
&= \frac{\phi(0) + A\Theta(\rho)}{A(A-1)} + \frac{\|\phi\|_J}{A}.
\end{aligned}$$

Consequently, for every $t \in J \cup I$, it holds

$$|x(t)| \leqslant \max\left\{ \frac{\phi(0) + A\Theta(\rho)}{A-1}, \frac{\phi(0) + A\Theta(\rho)}{A(A-1)} + \frac{\|\phi\|_J}{A} \right\} = Q(\rho).$$

Therefore, since $x \in \partial\mathbb{P}_\rho$, we finally obtain

$$\|x\|_{J\cup I} = \rho \leqslant Q(\rho),$$

which contradicts ($H_3$).

We can now apply Theorem 2.2 to obtain that the boundary value problem (1.1)–(1.2) has at least one positive solution $x$, such that

$$\|x\|_{J\cup I} < \rho.$$

Now we will justify why this solution is nonzero. If $\phi \neq 0$, then there exists $t_1 \in J$ such that $\phi(t_1) \neq 0$. Then, by (1.2), we have $Ax_0(t_1) - x_T(t_1) = \phi(t_1) \neq 0$, or $Ax(t_1) - x(T + t_1) \neq 0$, which means that $x \neq 0$. If $\phi = 0$, then, by hypothesis, there exists $t_0 \in I$ with $f(t_0, 0) \neq 0$. Thus the zero function is not a solution of Eq. (1.1).

We remind that, by Lemma 2.6, $x$ is a solution of the boundary value problem (1.1)–(1.2) if and only if $x = Lx$ . Therefore if $x$ is a positive solution of the boundary value problem (1.1)–(1.2), then, taking into account the formula of $L$ and the fact that $A > 1$, we easily conclude that

$$x(t) \geqslant \frac{\phi(0)}{A(A-1)}, \quad t \in J \cup I,$$

which implies that

$$\|x\|_{J\cup I} \geqslant \Phi.$$

Observe that $\Phi \leqslant Q(\rho)$ and hence, by ($H_3$), $\Phi < \rho$. Therefore we finally have

$$\Phi \leqslant \|x\|_{J\cup I} < \rho$$

and the proof is complete. □

In order to prove our second result we need the following assumption:

($H_4$) There exist $E \subseteq I$, with meas $E > 0$, and functions $u : I \to [0, r]$, continuous $v : E \to [0, +\infty)$ with $\sup\{v(t)\colon\ t \in E\} > 0$ and nondecreasing $w : [0, +\infty) \to [0, +\infty)$ such that

$$t - u(t) \geqslant 0, \quad t \in E,$$

and

$$f(t, y) \geqslant v(t)w\bigl(y\bigl(-u(t)\bigr)\bigr), \quad (t, y) \in E \times C^{+}(J).$$

Now we set

$$\mu := \frac{1}{A(A-1)} \int_E v(\theta)\, d\theta, \qquad \Lambda := \Phi + \frac{\phi(-r)}{A}$$

and we are ready to prove the following

**Theorem 3.2.** *Suppose that* ($H_1$)–($H_4$) *hold and, moreover, $\phi$ is a nondecreasing function. Also suppose that there exists $\gamma > 0$ such that*

$$\gamma \leqslant \frac{1}{A} w(\gamma)\mu. \tag{3.1}$$

*Then the boundary value problem* (1.1)–(1.2) *has at least one positive solution $x$, such that*

$$D \leqslant \|x\|_{J \cup I} < \max\{\tau, \rho\},$$

*where*

$$D = \begin{cases} \rho, & \text{if } \tau > \rho, \\ \max\{\tau, \Lambda\}, & \text{if } \tau < \rho, \end{cases}$$

*$\rho$ is the constant involved in* ($H_3$) *and $\tau := A\gamma \neq \rho$. More precisely, in any case we have*

$$x(t) \geqslant \Lambda, \quad t \in J \cup I.$$

**Proof.** Define the set

$$\mathbb{K} := \left\{ x \in C(J \cup I)\colon\ x \geqslant 0,\ x \text{ is nondecreasing and } x(0) \geqslant \frac{1}{A} x(T) \right\}.$$

Notice that $\mathbb{K}$ is a cone in $C(J \cup I)$. By ($H_1$) and ($H_2$), it is clear that, for any $x \in \bar{K}_d$, where $d = \max\{\rho, \tau\}$, we have $Lx(t) \geqslant 0$, $t \in J \cup I$. Also, since $x \geqslant 0$ we have, also by ($H_2$), that $(Lx)'(t) = f(t, x_t) \geqslant 0$, $t \in I$. Namely $Lx/I$ is a nondecreasing function. Also, taking into account the formula of $Lx/J$ and the fact that $\phi$ is a nondecreasing function, we derive easily that $Lx/J$ is also a nondecreasing function. Since $Lx$ is continuous at zero, we conclude that $Lx$ is nondecreasing on $J \cup I$. Moreover it is clear that $ALx(0) - Lx(T) = \phi(0)$. By ($H_1$), we have $\phi(0)/A \geqslant 0$ and thus

$$Lx(0) = \frac{1}{A} Lx(T) + \frac{\phi(0)}{A} \geqslant \frac{1}{A} Lx(T).$$

So $L : \bar{\mathbb{K}}_d \to \mathbb{K}$. Also, from ($H_2$) it follows that $L$ is a completely continuous operator.

Furthermore, as we did in Theorem 3.1, we can prove that $Lx \neq \lambda x$ for every $\lambda \geqslant 1$ and $x \in \partial\mathbb{K}_\rho$.

Now we will prove that $\|Lx\|_{J\cup I} \geqslant \|x\|_{J\cup I}$ for every $x \in \partial\mathbb{K}_\tau$. For this purpose it suffices to prove that $Lx(t) \geqslant \tau$ for every $x \in \partial\mathbb{K}_\tau$ and $t \in J \cup I$. By ($H_1$) we have

$$\begin{aligned}
Lx(-r) &= \frac{\phi(0)}{A(A-1)} + \frac{1}{A(A-1)}\int_0^T f(\theta, x_\theta)\,d\theta + \frac{1}{A}\int_0^{T-r} f(\theta, x_\theta)\,d\theta + \frac{\phi(-r)}{A} \\
&\geqslant \frac{1}{A(A-1)}\int_0^T f(\theta, x_\theta)\,d\theta.
\end{aligned}$$

So, using ($H_4$) and the fact that $w, x$ are nondecreasing, we obtain

$$\begin{aligned}
Lx(-r) &\geqslant \frac{1}{A(A-1)}\int_E v(\theta)w\big(x_\theta\big(-u(\theta)\big)\big)\,d\theta \\
&= \frac{1}{A(A-1)}\int_E v(\theta)w\big(x\big(\theta-u(\theta)\big)\big)\,d\theta \\
&\geqslant \frac{1}{A(A-1)}\int_E v(\theta)w\big(x(0)\big)\,d\theta.
\end{aligned}$$

But $x(0) \geqslant x(T)/A$, and $x(T) = \|x\|_{J\cup I}$, since $x$ is nondecreasing. Therefore, taking into account (3.1) we have

$$\begin{aligned}
Lx(-r) &\geqslant \frac{1}{A(A-1)}\int_E w\left(\frac{1}{A}\|x\|_{J\cup I}\right)v(\theta)\,d\theta \\
&\geqslant \frac{1}{A(A-1)}w\left(\frac{1}{A}\tau\right)\int_E v(\theta)\,d\theta \\
&= w(\gamma)\mu \geqslant A\gamma = \tau.
\end{aligned}$$

Hence, since $Lx$ is nondecreasing, we have

$$Lx(t) \geqslant \tau = \|x\|_{J\cup I}, \quad t \in J \cup I.$$

Therefore, for every $x \in \partial\mathbb{K}_\tau$ we have $\|Lx\|_{J\cup I} \geqslant \|x\|_{J\cup I}$.

Thus applying Theorem 2.4 we get that the boundary value problem (1.1)–(1.2) has at least one positive solution $x$, such that

$$\min\{\tau, \rho\} < \|x\|_{J\cup I} < \max\{\tau, \rho\}. \tag{3.2}$$

But $x$ is a positive and nondecreasing solution of the boundary value problem (1.1)–(1.2), which means that $x = Lx$ and it is easy to see that for every $t \in I$ we have

$$x(t) \geqslant x(-r) = Lx(-r) \geqslant \Lambda,$$

which implies that

$$\|x\|_{J\cup I} \geqslant \Lambda.$$

Then, in view of (3.2), we obtain

$$\max\bigl\{\min\{\tau,\rho\},\Lambda\bigr\} < \|x\|_{J\cup I} < \max\{\tau,\rho\}. \tag{3.3}$$

Now we observe that for every $\theta > 0$ we have $\Lambda \leqslant Q(\theta)$. So, since $Q(\rho) < \rho$, we have $\Lambda < \rho$ and if $\tau > \rho$, then $\max\{\min\{\tau,\rho\},\Lambda\} = \max\{\rho,\Lambda\} = \rho$. On the other hand, if $\tau < \rho$, then $\max\{\min\{\tau,\rho\},\Lambda\} = \max\{\tau,\Lambda\}$. Therefore (3.3) takes the form

$$D \leqslant \|x\|_{J\cup I} < \max\{\tau,\rho\}$$

and the proof is complete. □

Now, consider the following assumption ($H_5$), which is the analogue of assumption ($H_4$), when the function $w$ is nonincreasing.

($H_5$) There exist $E \subseteq I$, with $\operatorname{meas} E > 0$, and functions $u : I \to [0,r]$, continuous $v : E \to [0,+\infty)$ with $\sup\{v(t)\colon t \in E\} > 0$ and nonincreasing $w : [0,+\infty) \to [0,+\infty)$ such that

$$f(t,y) \geqslant v(t)w\bigl(y\bigl(-u(t)\bigr)\bigr), \quad (t,y) \in E \times C^+(J).$$

Then we have the following:

**Theorem 3.3.** *Suppose that* ($H_1$)–($H_3$), ($H_5$) *hold and, moreover, $\phi$ is a nondecreasing function. Also suppose that there exists $\tau > 0$ such that*

$$\tau \leqslant w(\tau)\mu, \tag{3.4}$$

*where $\Lambda$ is defined in Theorem* 3.2. *Then the boundary value problem* (1.1)–(1.2) *has at least one positive solution $x$, such that*

$$D \leqslant \|x\|_{J\cup I} < \max\{\tau,\rho\},$$

*where $D$ is defined in Theorem* 3.2, *$\rho$ is the constant involved in* ($H_3$) *and $\rho \neq \tau$. More precisely, in any case we have*

$$x(t) \geqslant \Lambda, \quad t \in J \cup I.$$

**Proof.** Define the set

$$\mathbb{K} := \bigl\{x \in C(J\cup I)\colon\ x \geqslant 0,\ x \text{ is nondecreasing}\bigr\}.$$

Notice that $\mathbb{K}$ is a cone in $C(J\cup I)$. By ($H_1$) and ($H_2$), it is clear that, for any $x \in \bar{K}_d$, where $d = \max\{\rho,\tau\}$, we have $Lx(t) \geqslant 0$, $t \in J \cup I$. Also, since $x \geqslant 0$ we have, also by ($H_2$), that $(Lx)'(t) = f(t,x_t) \geqslant 0$, $t \in I$. Namely $Lx/I$ is a nondecreasing function. Also, taking into account the formula of $Lx/J$ and the fact that $\phi$ is a nondecreasing function, we derive easily that $Lx/J$ is also a nondecreasing function. Since $Lx$ is continuous at zero, we conclude that $Lx$ is nondecreasing on $J \cup I$. So $L : \bar{\mathbb{K}}_d \to \mathbb{K}$. Also, from ($H_2$) it follows that $L$ is a completely continuous operator.

Furthermore, as we did in Theorem 3.1, we can prove that $Lx \neq \lambda x$ for every $\lambda \geqslant 1$ and $x \in \partial\mathbb{K}_\rho$.

Now we will prove that $\|Lx\|_{J\cup I} \geqslant \|x\|_{J\cup I}$ for every $x \in \partial\mathbb{K}_\tau$. For this purpose it suffices to prove that $Lx(t) \geqslant \tau$ for every $x \in \partial\mathbb{K}_\tau$ and $t \in J \cup I$. As in Theorem 3.1, using $(H_1)$ and $(H_5)$, we obtain

$$Lx(-r) \geqslant \frac{1}{A(A-1)} \int_E v(\theta) w\bigl(x\bigl(\theta - u(\theta)\bigr)\bigr)\, d\theta.$$

But $0 \leqslant x(t) \leqslant \tau$ for any $t \in J \cup I$. Therefore, taking into account the fact that $w$ is nonincreasing and (3.4), we have

$$Lx(-r) \geqslant \frac{1}{A(A-1)} w(\tau) \int_E v(\theta)\, d\theta = w(\tau)\mu \geqslant \tau.$$

Hence, since $Lx$ is nondecreasing, we have

$$Lx(t) \geqslant \tau = \|x\|_{J\cup I}, \quad t \in J \cup I.$$

Therefore, for every $x \in \partial\mathbb{K}_\tau$ we have $\|Lx\|_{J\cup I} \geqslant \|x\|_{J\cup I}$.

So, applying Theorem 2.4 we get that there exists at least one positive solution $x$ of the boundary value problem (1.1)–(1.2), such that (3.2) holds. The rest of the proof is similar to that of Theorem 3.1 and evidently the proof is complete. □

Combining Theorems 3.1 and 3.2 (respectively 3.3) we can prove easily the following theorem, which ensures the existence of two positive solutions for the boundary value problem (1.1)–(1.2).

**Theorem 3.4.** *Suppose that* $(H_1)$–$(H_4)$ (*respectively* $(H_1)$–$(H_3)$, $(H_5)$) *hold and, moreover, $\phi$ is a nondecreasing function. Also, suppose that if $\phi = 0$, there exists $t_0 \in I$ such $f(t_0, 0) \neq 0$ and, additionally, there exists $\gamma > 0$ such that* (3.1) (*respectively* (3.4)) *holds. Then, if $\rho < A\gamma$* (*respectively $\rho < \gamma$*)*, the boundary value problem* (1.1)–(1.2) *has at least two positive solutions $x_1, x_2$ such that*

$$\Lambda \leqslant \|x_1\|_{J\cup I} < \rho < \|x_2\|_{J\cup I} < \tau,$$

*where*

$$\tau := \begin{cases} A\gamma, & \text{if } (H_4) \text{ holds}, \\ \gamma, & \text{if } (H_5) \text{ holds}. \end{cases}$$

*Moreover we have*

$$x_i(t) \geqslant \Lambda, \quad t \in J \cup I,\ i = 1, 2.$$

Going a step further, we have the following theorem about the existence of countable positive solutions of the boundary value problem (1.1)–(1.2), which is a direct consequence of the above Theorems 3.2 and 3.3.

**Theorem 3.5.** *Assume that* $(H_1)$, $(H_2)$, $(H_4)$ (*respectively* $(H_1)$, $(H_2)$, $(H_5)$) *hold, $\phi$ is a nondecreasing function and there exist two strictly increasing real sequences* $(\rho_\nu)_{\nu\in\mathbb{N}}$, $(\gamma_\nu)_{\nu\in\mathbb{N}}$ ($\mathbb{N}$ *the set of natural numbers*) *such that*

$$\rho_\nu < \tau_\nu := A\gamma_\nu < \rho_{\nu+1}, \quad \nu \in \mathbb{N}$$

$$(\textit{respectively} \quad \rho_\nu < \tau_\nu := \gamma_\nu < \rho_{\nu+1}, \quad \nu \in \mathbb{N}).$$

*Moreover, assume that* (H$_3$) *is satisfied for all* $\rho_\nu$ *in place of* $\rho$ *and* (3.1) (*respectively* (3.4)) *is also satisfied for all* $\gamma_\nu$ *in place of* $\gamma$. *Then, the boundary value problem* (1.1)–(1.2) *has a sequence of positive solutions* $(x_\nu)_{\nu\in\mathbb{N}}$ *such that*

$$\rho_\nu < \|x_\nu\|_{J\cup I} < \tau_\nu < \|x_{\nu+1}\|_{J\cup I} < \rho_{\nu+1}, \quad \nu \in \mathbb{N}.$$

*Moreover we have*

$$x_\nu(t) \geqslant \Lambda, \quad t \in J \cup I, \ \nu \in \mathbb{N}.$$

**Remark 3.6.** It is clear that the assumption

there exists $\gamma > 0$ such that (3.1) (respectively (3.4)) holds

in Theorem 3.4, can be replaced by the following:

$$\limsup_{\theta\to+\infty} \frac{w(\theta)}{\theta} > \frac{A}{\mu} \qquad \left(\text{respectively} \quad \limsup_{\theta\to+\infty} \frac{w(\theta)}{\theta} > \frac{1}{\mu}\right).$$

Indeed if

$$\limsup_{\theta\to+\infty} \frac{w(\theta)}{\theta} > \frac{A}{\mu} \qquad \left(\text{respectively} \quad \limsup_{\theta\to+\infty} \frac{w(\theta)}{\theta} > \frac{1}{\mu}\right),$$

then there exists $\gamma > \rho/A$ (respectively $\gamma > \rho$) such that $w(\gamma)/\gamma > A/\mu$ (respectively $w(\gamma)/\gamma > 1/\mu$).

## 4. The ordinary case

If we choose $r = 0$ then we no longer have a functional boundary value problem, but an ordinary one instead. In this case, the boundary value problem is formed as follows:

$$x'(t) = f\big(t, x(t)\big), \quad t \in I, \tag{4.1}$$

$$Ax(0) - x(T) = C, \tag{4.2}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $A, C$ are real numbers satisfying the following:

($\hat{\mathrm{H}}_1$) $A > 1$ and $C \geqslant 0$.

**Definition 4.1.** A solution of the boundary value problem (4.1)–(4.2) is a function $x \in C(I)$ continuously differentiable on $I$, which satisfies equations (4.1)–(4.2). Additionally, $x$ is called positive solution if $x(t) \geqslant 0$, $t \in I$.

It is clear that a function $x \in C(I)$ is a solution of the boundary value problem (4.1)–(4.2) if and only if it satisfies the equation $x = \hat{L}x$, where the operator $\hat{L}: C(I) \to C(I)$ is given by the formula

$$\hat{L}x(t) := \frac{C}{A-1} + \frac{1}{A-1}\int_0^T f\big(\theta, x(\theta)\big)\,d\theta + \int_0^t f\big(\theta, x(\theta)\big)\,d\theta, \quad t \in I.$$

In this, ordinary, case assumptions ($H_2$)–($H_3$) are replaced by the following:

($\hat{H}_2$) Assume that $f(I \times [0,+\infty)) \subset [0,+\infty)$ and for every $t \in I$ the function $f(t,\cdot): [0,+\infty) \to [0,+\infty)$ maps bounded subsets of $[0,+\infty)$ into bounded subsets of $[0,\infty)$.

It is obvious that under assumption ($\hat{H}_2$), for every $s \in I$ and $m > 0$, $\sup_{y\in[0,m]} f(s,y)$ exists in $\mathbb{R}$. Then we set

$$\hat{F}(s,m) =: \sup_{y\in[0,m]} f(s,y).$$

Also for any $m > 0$ we set

$$\hat{\Theta}(m) := \max_{t\in I} \int_0^t \hat{F}(s,m)\,ds = \int_0^T \hat{F}(s,m)\,ds$$

and

$$\hat{Q}(m) := \frac{C + A\hat{\Theta}(m)}{A-1}.$$

($\hat{H}_3$) There exists $\rho > 0$ such that $\hat{Q}(\rho) < \rho$.

Now, observe that the analogues of assumptions ($H_4$), ($H_5$), for the ordinary case, can be unified in the following:

($\hat{H}_4$) There exist $E \subseteq I$, with meas $E > 0$, and functions $v: E \to [0,+\infty)$ continuous with $\sup\{v(t):\ t \in E\} > 0$, and $w: [0,+\infty) \to [0,+\infty)$ monotonous, such that

$$f(t,y) \geqslant v(t)w(y), \quad (t,y) \in E \times [0,+\infty).$$

Finally we set

$$\hat{\Phi} := \frac{C}{A-1}$$

and then we have the following theorems, which correspond to Theorems 3.1–3.5, for the ordinary case. The proofs of these theorems are omitted, since they can be easily derived from the proofs of Theorems 3.1–3.5, with some obvious modifications. Also it is easy to see that the analogue of the constant $\Lambda$ in the present ordinary case identifies with the constant $\hat{\Phi}$.

**Theorem 4.2.** *Suppose that* $(\hat{\mathrm{H}}_1)$–$(\hat{\mathrm{H}}_3)$ *hold. Also suppose that, in case* $C = 0$, *there exists* $t_0 \in I$ *such that* $f(t_0, 0) \neq 0$. *Then the boundary value problem* (4.1)–(4.2) *has at least one positive nonzero solution* $x$, *such that*

$$\hat{\Phi} \leqslant \|x\|_I < \rho.$$

*More precisely we have*

$$x(t) \geqslant \hat{\Phi}, \quad t \in I.$$

**Theorem 4.3.** *Suppose that* $(\hat{\mathrm{H}}_1)$–$(\hat{\mathrm{H}}_4)$ *hold. Also suppose that there exists* $\gamma > 0$ *such that*

$$\gamma < \frac{1}{A} w(\gamma)\mu, \quad \text{if } w \text{ is nondecreasing}, \tag{4.3}$$

*or*

$$\gamma < w(\gamma)\mu, \quad \text{if } w \text{ is nonincreasing}. \tag{4.4}$$

*Then the boundary value problem* (4.1)–(4.2) *has at least one positive solution* $x$, *such that*

$$\hat{D} \leqslant \|x\|_I < \max\{\tau, \rho\},$$

*where*

$$\hat{D} = \begin{cases} \rho, & \text{if } \tau > \rho, \\ \max\{\tau, \hat{\Phi}\}, & \text{if } \tau < \rho, \end{cases}$$

$\rho$ *is the constant involved in* $(\hat{\mathrm{H}}_4)$, $\rho \neq \tau$ *and* $\tau := A\gamma$, *if* $w$ *in nondecreasing, or* $\tau := \gamma$, *if* $w$ *in nonincreasing. More precisely in any case we have*

$$x(t) \geqslant \hat{\Phi}, \quad t \in I.$$

**Theorem 4.4.** *Suppose that conditions* $(\hat{\mathrm{H}}_1)$–$(\hat{\mathrm{H}}_4)$ *hold. Also, suppose that if* $C = 0$, *there exists* $t_0 \in I$ *such that* $f(t_0, 0) \neq 0$ *and, additionally, there exists* $\gamma > 0$ *such that* (4.3) *holds. Then, if* $\rho < \tau$, *the boundary value problem* (4.1)–(4.2) *has at least two positive solutions* $x_1, x_2$ *such that*

$$\hat{\Phi} \leqslant \|x_1\|_I < \rho < \|x_2\|_I < \tau,$$

*where* $\tau := A\gamma$, *if* $w$ *in nondecreasing, or* $\tau := \gamma$, *if* $w$ *in nonincreasing. Moreover we have*

$$x_i(t) \geqslant \hat{\Phi}, \quad t \in I, \ i = 1, 2.$$

**Theorem 4.5.** *Assume that* $(\hat{\mathrm{H}}_1)$, $(\hat{\mathrm{H}}_2)$, $(\hat{\mathrm{H}}_4)$ *hold and there exist two strictly increasing real sequences* $(\rho_\nu)_{\nu \in \mathbb{N}}$, $(\gamma_\nu)_{\nu \in \mathbb{N}}$ ($\mathbb{N}$ *the set of natural numbers*) *such that*

$$\rho_\nu < \tau_\nu := A\gamma_\nu < \rho_{\nu+1}, \quad \nu \in \mathbb{N}, \quad \text{if } w \text{ is nondecreasing},$$

*or*

$$\rho_\nu < \tau_\nu := \gamma_\nu < \rho_{\nu+1}, \quad \nu \in \mathbb{N}, \quad \text{if } w \text{ is nonincreasing}.$$

*Moreover, assume that* $(\hat{\mathrm{H}}_3)$ *is satisfied for all* $\rho_\nu$ *in place of* $\rho$ *and* (4.3)*, if* $w$ *is nondecreasing, or* (4.4)*, if* $w$ *is nonincreasing, is also satisfied for all* $\gamma_\nu$ *in place of* $\gamma$*. Then, the boundary value problem* (4.1)–(4.2) *has a sequence of positive solutions* $(x_\nu)_{\nu\in\mathbb{N}}$ *such that*

$$\rho_\nu < \|x_\nu\|_I < \tau_\nu < \|x_{\nu+1}\|_I < \rho_{\nu+1}, \quad \nu \in \mathbb{N}.$$

*Moreover we have*

$$x_\nu(t) \geqslant \hat{\Phi}, \quad t \in I, \ \nu \in \mathbb{N}.$$

## 5. Applications

**(1)** Consider the homogeneous functional boundary value problem

$$x'(t) = (\sin t)\sqrt{x\left(t - \frac{1}{4}\right) + 3}, \quad t \in I := [0,1], \tag{5.1}$$

$$5x_0 - 2x_1 = 0. \tag{5.2}$$

Here we have $f(t,y) = (\sin t)\sqrt{y(-1/4)+3}$, $t \in I$, $y \in C([-1/4,0])$, $A = 5/2$ and we choose $F(t,m) = \sqrt{m+3}\sin t$, $t \in I$, $m > 0$. So $\Theta(m) = (1-\cos 1)\sqrt{m+3}$, $m > 0$.

Now observe that assumptions $(\mathrm{H}_1)$–$(\mathrm{H}_2)$ hold and assumption $(\mathrm{H}_3)$ is satisfied for $\rho = 2$, since it is easy to see that $2 > (\sqrt{5}/3)(1-\cos 1) = Q(1)$. So we can apply Theorem 3.1 to get that the boundary value problem (5.1)–(5.2) has at least one nonzero positive solution $x$ such that

$$\|x\|_{[-1/4,1]} < 2.$$

Also we have

$$f(t,y) \geqslant v(t)w\big(y\big(-u(t)\big)\big), \quad (t,y) \in \left[\frac{1}{4}, 1\right] \times C\left(\left[-\frac{1}{4}, 0\right]\right),$$

where $v(t) = \sin t$, $t \in I$, $w(t) = \sqrt{t+3}$, $t \in \mathbb{R}^+$, and $u(t) = 1/4$, $t \in I$. Obviously $\sup\{v(t)\colon t \in [1/4,1]\} > 0$. Therefore for $E = [1/4,1]$, assumption $(\mathrm{H}_4)$ is also satisfied.

Moreover we have

$$\mu = \frac{4}{15}\left(\cos\frac{1}{4} - \cos 1\right),$$

$\Lambda = 0$ and inequality (3.1) takes the form

$$\frac{\gamma}{\sqrt{\gamma+3}} \leqslant \frac{8}{75}\left(\cos\frac{1}{4} - \cos 1\right),$$

which holds for $\gamma = 0.02$. So, all requirements of Theorem 3.2 are fulfilled and hence the boundary value problem (5.1)–(5.2) has at least one positive solution $x$, such that

$$\frac{5}{2}\gamma = 0.05 < \|x\|_{[-1/4,1]} < 2.$$

**(2)** Consider the ordinary boundary value problem

$$x'(t) = \sqrt{t}\,\frac{x^2(t)+1}{7}, \quad t \in [0,1], \tag{5.3}$$

$$7x(0) - 3x(1) = 0. \tag{5.4}$$

Here we have $f(t,y) = \sqrt{t}(y^2+1)/7$, $t \in [0,1]$, $y \in \mathbb{R}$, $A = 7/3$ and we choose $\hat{F}(s,m) = \sqrt{s}(m^2+1)/7$, $s \in [0,1]$, $m > 0$ and $\hat{\Theta}(m) = (2/21)(m^2+1)$, $m > 0$.

Now, observe that assumptions $(\hat{\mathrm{H}}_1)$–$(\hat{\mathrm{H}}_2)$ hold and assumption $(\hat{\mathrm{H}}_3)$ is satisfied for $\rho = 1$, since it holds

$$\rho > \frac{1}{6}(\rho^2+1).$$

Also, for $t = 1/2$ and any $y \in C([0,1])$, it holds $f(1/2, y) > 0$, so Theorem 4.2 can be applied to prove that the boundary value problem (5.3)–(5.4) has at least one positive nonzero solution $x_1$ such that

$$\|x_1\|_{[0,1]} < 1.$$

Also we obtain

$$f(t,y) \geqslant v(t)w(y),$$

where $v(t) = \sqrt{t}$, $t \in [0,1]$ and $w(t) = (t^2+1)/7$, $t \in \mathbb{R}^+$.

Obviously $\sup\{v(t)\colon t \in [0,1]\} = 1 > 0$, $\hat{\Phi} = 0$, and

$$\gamma \leqslant \frac{1}{A} w(\gamma)\mu = \frac{9}{686}(\gamma^2+1)$$

is satisfied for

$$\gamma \geqslant \frac{686 + \sqrt{470272}}{18}.$$

Consequently, if we choose $\rho = 1$ and $\tau = 7\gamma/3$, Theorem 4.3 applies and we conclude that the boundary value problem (5.3)–(5.4) has at least one positive solution $x_2$, such that

$$1 < \|x_2\|_{[0,1]} < \frac{7}{3}\gamma.$$

Finally, comparing the norms of the solutions $x_1, x_2$ we conclude that $\|x_1\|_{[0,1]} \neq \|x_2\|_{[0,1]}$. So $x_1 \neq x_2$, which means that the boundary value problem (5.3)–(5.4) has at least two positive and nonzero solutions, such that

$$0 < \|x_1\|_{[0,1]} < 1 < \|x_2\|_{[0,1]} < \frac{7}{3}\gamma.$$

**(3)** Consider the ordinary boundary value problem

$$x'(t) = tx(t), \quad t \in I, \tag{5.5}$$

$$4x(0) - x(1) = \frac{1}{2}. \tag{5.6}$$

Here we have $\hat{F}(s,m) = sm$, $s \in I$, $m \in (0,+\infty)$ and $\hat{\Theta}(m) = m/2$. So assumptions of Theorem 4.2 are satisfied for $\rho = 1$, since

$$1 > \frac{5}{6}.$$

Hence the boundary value problem (5.5)–(5.6) has at least one positive solution $x$ such that

$$\frac{1}{6} < \|x\|_I < 1.$$

Indeed, it easy to see that

$$x(t) = \frac{e^{t^2/2}}{2(4-\sqrt{e})}, \quad t \in I,$$

is a positive solution of the boundary value problem (5.5)–(5.6). Now observe that

$$\frac{1}{6} \leqslant \frac{\sqrt{e}}{2(4-\sqrt{e})} \leqslant 1,$$

which confirms the results of Theorem 4.2.

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